A new hierarchy of integrable differential-difference equations and Darboux transformation

1998 J. Phys. A: Math. Gen. 31 L677

(http://iopscience.iop.org/0305-4470/31/38/004)

J. Phys. A: Math. Gen. **31** (1998) L677–L684. Printed in the UK PII: S0305-4470(98)95519-6

LETTER TO THE EDITOR

A new hierarchy of integrable differential-difference equations and Darboux transformation

Yongtang Wu† and Xianguo Geng‡

† Department of Computing Studies, Hong Kong Baptist University, 224 Waterloo Road, Kowloon, Hong Kong, People's Republic of China

‡ CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China and Department of Mathematics, Zhengzhou University, Zhengzhou, Henan 450052, People's Republic of China

Received 29 June 1998

Abstract. A new discrete isospectral problem and the corresponding hierarchy of nonlinear differential-difference equations are proposed. It is shown that the hierarchy of differential-difference equations possesses the Hamiltonian structures. A Darboux transformation for the discrete spectral problem is found. As an application, two-soliton solutions for the first system of differential-difference equations in the hierarchy are given.

The study of integrable differential-difference systems has aroused increasing interest in the last few years [1–4]. A key feature of integrable differential-difference equations is the fact that they can be expressed as the compatibility condition of two linear discrete spectral problems, i.e. a Lax pair, which plays a crucial role in the inverse scattering transformation (IST) [1, 5] and Darboux transformation (DT) [6–8]. A major difficulty, however, is the problem of associating nonlinear differential-difference equations with appropriate spectral problems. Therefore it interests us to search for a new discrete spectral problem and the corresponding nonlinear differential-difference equations.

In this letter we first introduce a discrete spectral problem

$$y_{n+1} = U_n y_n \qquad U_n = \begin{pmatrix} 1+\lambda u_n v_n & u_n \\ \lambda v_n & 1 \end{pmatrix} \tag{1}$$

where u_n and v_n are two potentials, λ is a constant spectral parameter, and derive the corresponding hierarchy of nonlinear differential-difference equations. The first system of nonlinear differential-difference equations in the hierarchy is as follows

$$u_{nt} = \frac{1}{v_{n+1}} - \frac{1}{v_n} \qquad v_{nt} = \frac{1}{u_n} - \frac{1}{u_{n-1}}. \tag{2}$$

Then using the gauge transformation, we establish a DT of spectral problem (1). As an application, two-soliton solutions of equations (1) are given.

In order to derive a hierarchy of differential-difference equations associated with (1), we first solve the stationary discrete zero-curvature equation:

$$V_{n+1}U_n - U_n V_n = 0 \qquad V_n = \begin{pmatrix} A_n & \lambda^{-1}B_n \\ C_n & -A_n \end{pmatrix}. \tag{3}$$

It is easy to see that (3) is equivalent to

$$\begin{aligned}&\Delta B_n - \lambda u_n(v_n B_n - A_{n+1} - A_n) = 0\\&-\Delta^* C_{n+1} + \lambda v_n(u_n C_{n+1} - A_{n+1} - A_n) = 0\\&\Delta A_n = u_n C_{n+1} - v_n B_n\end{aligned}\tag{4}$$

where $\Delta h_n = h_{n+1} - h_n$, $\Delta^* h_n = h_{n-1} - h_n$. Substituting the following expansions

$$A_n = \sum_{j\geqslant 0} A_n^{(j)}\lambda^{-j} \qquad B_n = \sum_{j\geqslant 0} B_n^{(j)}\lambda^{-j} \qquad C_n = \sum_{j\geqslant 0} C_n^{(j)}\lambda^{-j}\tag{5}$$

into (4), we arrive at

$$\begin{aligned}&v_n B_n^{(0)} - A_{n+1}^{(0)} - A_n^{(0)} = 0\\&u_n C_{n+1}^{(0)} - A_{n+1}^{(0)} - A_n^{(0)} = 0\\&\Delta A_n^{(0)} = u_n C_{n+1}^{(0)} - v_n B_n^{(0)}\end{aligned}\tag{6}$$

and

$$\begin{aligned}&\Delta B_n^{(j-1)} = u_n(v_n B_n^{(j)} - A_{n+1}^{(j)} - A_n^{(j)})\\&-\Delta^* C_{n+1}^{(j-1)} = v_n(A_{n+1}^{(j)} + A_n^{(j)} - u_n C_{n+1}^{(j)})\\&\Delta A_n^{(j)} = u_n C_{n+1}^{(j)} - v_n B_n^{(j)} \qquad j \geqslant 1.\end{aligned}\tag{7}$$

The above recursion equations can be solved successively to deduce the results

$$A_n^{(0)} = \tfrac{1}{2}\alpha_0(\text{constant}) \qquad B_n^{(0)} = \alpha_0 v_n^{-1} \qquad C_n^{(0)} = \alpha_0 u_{n-1}^{-1}$$

$$A_n^{(1)} = -\frac{\alpha_0}{u_{n-1}v_n} \qquad B_n^{(1)} = -\frac{\alpha_0}{v_n^2}\left(\frac{1}{u_{n-1}} + \frac{1}{u_n}\right) \qquad C_n^{(1)} = -\frac{\alpha_0}{u_{n-1}^2}\left(\frac{1}{v_{n-1}} + \frac{1}{v_n}\right).$$

Equation (7) can be rewritten as

$$K_n G_n^{(j-1)} = J_n G_n^{(j)} \qquad G_n^{(0)} = \alpha_0(u_n^{-1}, v_n^{-1})^T\tag{8}$$

with $G_n^{(j)} = (C_{n+1}^{(j)}, B_n^{(j)})^T$. Here K_n and J_n are two skew-symmetric operators

$$K_n = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix} \qquad J_n = \begin{pmatrix} -u_n^2 - 2u_n\Delta^{-1}u_n & 2u_n v_n + 2u_n\Delta^{-1}v_n \\ 2v_n\Delta^{-1}u_n & -v_n^2 - 2v_n\Delta^{-1}v_n \end{pmatrix}.$$

In fact, a direct calculation shows that

$$\langle Kf, g\rangle = -\langle f, Kg\rangle \qquad \langle Jf, g\rangle = -\langle f, Jg\rangle\tag{9}$$

in view of equality

$$\sum_{n\in\mathbb{Z}} h_n\Delta^{-1}l_n = -\sum_{n\in\mathbb{Z}}(h_n l_n + l_n\Delta^{-1}h_n)\tag{10}$$

where h_n and l_n are scalar functions, $f_n = (f_n^{(1)}, f_n^{(2)})^T$, $g_n = (g_n^{(1)}, g_n^{(2)})^T$

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}}\sum_{i=1}^{2} f_n^{(i)} g_n^{(i)} \qquad \Delta^{-1}h_n = \tfrac{1}{2}\Bigg(\sum_{j\leqslant n-1} - \sum_{j\geqslant n}\Bigg)h_j.$$

This means that the operators K_n and J_n are skew-symmetric. Let y_n satisfy the eigenvalue problem (1) and the auxiliary problem

$$y_{nt} = V_n^{(m)} y_n \qquad V_n^{(m)} = \lambda\begin{pmatrix} \mathcal{A}_n^{(m)} & \lambda^{-1}\mathcal{B}_n^{(m)} \\ \mathcal{C}_n^{(m)} & -\mathcal{A}_n^{(m)} \end{pmatrix}\tag{11}$$

with

$$\mathcal{A}_n^{(m)} = \sum_{j=0}^{m} A_n^{(j)} \lambda^{m-j} \qquad \mathcal{B}_n^{(m)} = \sum_{j=0}^{m} B_n^{(j)} \lambda^{m-j} \qquad \mathcal{C}_n^{(m)} = \sum_{j=0}^{m} C_n^{(j)} \lambda^{m-j}.$$

Then the compatibility condition between (1) and (11) yields a discrete zero-curvature equation, $U_{nt} + U_n V_n^{(m)} - V_{n+1}^{(m)} U_n = 0$, which is equivalent to the hierarchy of nonlinear differential-difference equations

$$(u_{nt}, v_{nt})^T = K_n G_n^{(m)} \qquad m \geqslant 0. \tag{12}$$

As $\alpha_0 = 1$, the first two systems of differential-difference equations in the hierarchy (12) are equations (2) and the following:

$$\begin{aligned} u_{nt} &= \frac{1}{v_n^2}\left(\frac{1}{u_{n-1}} + \frac{1}{u_n}\right) - \frac{1}{v_{n+1}^2}\left(\frac{1}{u_n} + \frac{1}{u_{n+1}}\right) \\ v_{nt} &= \frac{1}{u_{n-1}^2}\left(\frac{1}{v_n} + \frac{1}{v_{n-1}}\right) - \frac{1}{u_n^2}\left(\frac{1}{v_{n+1}} + \frac{1}{v_n}\right). \end{aligned} \tag{13}$$

To establish the Hamiltonian structures of differential-difference equations (12), we apply the trace identity [3]:

$$\left(\frac{\delta}{\delta u_n}, \frac{\delta}{\delta v_n}\right) \operatorname{tr}\left(\hat{V}_n \frac{\partial U_n}{\partial \lambda}\right) = \left[\lambda^{-\varepsilon}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\varepsilon}\right]\left(\operatorname{tr}\left(\hat{V}_n \frac{\partial U_n}{\partial u_n}\right), \operatorname{tr}\left(\hat{V}_n \frac{\partial U_n}{\partial v_n}\right)\right) \tag{14}$$

where $\hat{V}_n = V_n U_n^{-1}$, tr means trace of a matrix, ε is a constant to be fixed. It is easy to calculate that

$$\begin{aligned} &\operatorname{tr}\left(\hat{V}_n \frac{\partial U_n}{\partial \lambda}\right) = \lambda^{-1} v_n B_n \qquad \operatorname{tr}\left(\hat{V}_n \frac{\partial U_n}{\partial u_n}\right) = 2\lambda v_n A_n - \lambda v_n^2 B_n + C_n \\ &\operatorname{tr}\left(\hat{V}_n \frac{\partial U_n}{\partial v_n}\right) = B_n. \end{aligned} \tag{15}$$

Noticing (4), we have

$$2\lambda v_n A_n - \lambda v_n^2 B_n + C_n = C_{n+1}. \tag{16}$$

Substituting (15) and (16) into (14) and comparing coefficients for the same power of λ, we obtain

$$(\delta/\delta u_n, \delta/\delta v_n) v_n B_n^{(j)} = (\varepsilon - j)(C_{n+1}^{(j)}, B_n^{(j)}) \tag{17}$$

and $\varepsilon(u_n^{-1}, v_n^{-1}) = 0$ which implies $\varepsilon = 0$. Hence we have

$$(\delta/\delta u_n, \delta/\delta v_n)^T H_j = G_n^{(j)} \qquad H_j = -\frac{1}{j} v_n B_n^{(j)} \qquad j \geqslant 1. \tag{18}$$

It is not difficult to verify that

$$(\delta/\delta u_n, \delta/\delta v_n)^T H_0 = G_n^{(0)} \qquad H_0 = \alpha_0 \ln u_n v_n. \tag{19}$$

Therefore we get the Hamiltonian form of the hierarchy (12)

$$\begin{pmatrix} u_{nt} \\ v_{nt} \end{pmatrix} = K_n G_n^{(m)} = K_n \begin{pmatrix} \delta/\delta u_n \\ \delta/\delta v_n \end{pmatrix} H_m \qquad m \geqslant 0. \tag{20}$$

In what follows, we shall construct the DT of eigenvalue problem (1). Let $\phi_n = (\phi_n^1, \phi_n^2)^T$, $\psi_n = (\psi_n^1, \psi_n^2)^T$ be two basic solutions of (1) and use (ϕ_n, ψ_n) to define a 2×2 matrix T_n by

$$T_n = \begin{pmatrix} 1 + \lambda a_n & b_n \\ \lambda c_n & 1 + \lambda d_n \end{pmatrix} \tag{21}$$

with

$$a_n = \frac{\alpha_1(n) - \alpha_2(n)}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)} \qquad b_n = \frac{\lambda_2 - \lambda_1}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}$$
$$c_n = \frac{(\lambda_2 - \lambda_1)\alpha_1(n)\alpha_2(n)}{\lambda_1\lambda_2(\alpha_1(n) - \alpha_2(n))} \qquad d_n = \frac{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}{\lambda_1\lambda_2(\alpha_1(n) - \alpha_2(n))} \tag{22}$$

$$\alpha_i(n) = \frac{\phi_n^2(\lambda_i) - \gamma_i\psi_n^2(\lambda_i)}{\phi_n^1(\lambda_i) - \gamma_i\psi_n^1(\lambda_i)} \qquad i = 1, 2 \tag{23}$$

where parameters λ_i and γ_i ($\lambda_1 \neq \lambda_2, \gamma_1 \neq \gamma_2$) are suitably chosen such that all the denominators in (22) and (23) are not zero. From (1) and (23) we have

$$\alpha_i(n+1) = \mu_i(n)/\nu_i(n) \qquad i = 1, 2 \tag{24}$$

with

$$\mu_i(n) = \lambda_i v_n + \alpha_i(n) \qquad \nu_i(n) = 1 + \lambda_i u_n v_n + u_n\alpha_i(n).$$

Using (22) and (24), we have

$$a_{n+1} = \frac{\mu_1(n)\nu_2(n) - \mu_2(n)\nu_1(n)}{\lambda_1\mu_2(n)\nu_1(n) - \lambda_2\mu_1(n)\nu_2(n)} \qquad b_{n+1} = \frac{(\lambda_2 - \lambda_1)\nu_1(n)\nu_2(n)}{\lambda_1\mu_2(n)\nu_1(n) - \lambda_2\mu_1(n)\nu_2(n)}$$
$$c_{n+1} = \frac{(\lambda_2 - \lambda_1)\mu_1(n)\mu_2(n)}{\lambda_1\lambda_2(\mu_1(n)\nu_2(n) - \mu_2(n)\nu_1(n))} \qquad d_{n+1} = \frac{\lambda_1\mu_2(n)\nu_1(n) - \lambda_2\mu_1(n)\nu_2(n)}{\lambda_1\lambda_2(\mu_1(n)\nu_2(n) - \mu_2(n)\nu_1(n))}. \tag{25}$$

Through tedious calculations, we can verify from (25) and (22) that the equalities

$$\Delta a_n + v_n b_n - c_{n+1}(u_n + \Delta b_n) = 0 \tag{26}$$
$$(u_n c_{n+1} + d_{n+1})(u_n + \Delta b_n) - u_n a_{n+1} = 0 \tag{27}$$
$$(a_n - v_n b_n)(v_n + \Delta c_n) - v_n d_n = 0. \tag{28}$$

Noticing (22) and (21), we have

$$\det T_n = \frac{1}{\lambda_1\lambda_2}(\lambda - \lambda_1)(\lambda - \lambda_2). \tag{29}$$

Now we introduce a gauge transformation

$$\hat{y}_n = T_n y_n \tag{30}$$

which transforms (1) into an eigenvalue problem of $\hat{y}_n$ in the case $\lambda \neq \lambda_i (i = 1, 2)$ as follows

$$\hat{y}_{n+1} = \hat{U}_n \hat{y}_n \tag{31}$$

with

$$\hat{U}_n = T_{n+1} U_n T_n^{-1}. \tag{32}$$

It turns out that $\lambda = \lambda_i (i = 1, 2)$ are removable isolated singularities of $\hat{U}_n$ (see (33) below). Thus we can define $\hat{U}_n$ for all λ by analytic continuation.

Proposition 1. The matrix $\hat{U}_n$ determined by (32) has the same form as U_n:

$$\hat{U}_n = \begin{pmatrix} 1 + \lambda\hat{u}_n\hat{v}_n & \hat{u}_n \\ \lambda\hat{v}_n & 1 \end{pmatrix} \tag{33}$$

where the transformation formulae from old potentials u_n, v_n into new ones are as follows

$$\hat{u}_n = u_n + \Delta b_n \qquad \hat{v}_n = v_n + \Delta c_n. \tag{34}$$

The transformation (30) and (34): $(y_n; u_n, v_n) \to (\hat{y}_n; \hat{u}_n, \hat{v}_n)$ is usually called a DT of eigenvalue problem (1).

Proof. Let $T_n^{-1} = T_n^*/\det T_n$ and

$$T_{n+1}U_nT_n^* = \begin{pmatrix} f_{11}(\lambda, n) & f_{12}(\lambda, n) \\ f_{21}(\lambda, n) & f_{22}(\lambda, n) \end{pmatrix}. \tag{35}$$

It is easy to see that $f_{11}(\lambda, n)$, $f_{21}(\lambda, n)$ or $f_{12}(\lambda, n)$, $f_{22}(\lambda, n)$ are first-order polynomials in λ or zero-order polynomial in λ, respectively. From (22), we obtain

$$1 + \lambda_i a_n = -b_n\alpha_i(n) \qquad 1 + \lambda_i d_n = -\lambda_i c_n\alpha_i^{-1}(n) \qquad i = 1, 2. \tag{36}$$

By using (36) and (23), it can be verified that λ_1 and λ_2 are roots of $f_{ij}(\lambda, n)$, $i, j = 1, 2$. Therefore, noticing (29) we have

$$T_{n+1}U_nT_n^* = (\det T_n)(P_n^{(0)} + \lambda P_n^{(1)}) \qquad P_n^{(1)} = \begin{pmatrix} p_{11}^{(1)}(n) & 0 \\ p_{21}^{(1)}(n) & 0 \end{pmatrix} \tag{37}$$

where $P_n^{(0)}$ and $P_n^{(1)}$ are independent of λ. Equation (37) can be written as

$$T_{n+1}U_n = (P_n^{(0)} + \lambda P_n^{(1)})T_n. \tag{38}$$

Equating the coefficients of λ and λ^0 in (38), we find

$$\begin{aligned} &P_n^{(0)} = \begin{pmatrix} 1 & \hat{u}_n \\ 0 & 1 \end{pmatrix} \\ &p_{11}^{(1)}(n) = u_nv_n + v_nb_{n+1} + \Delta a_n - (u_n + \Delta b_n)c_n \qquad p_{21}^{(1)}(n) = \hat{v}_n. \end{aligned} \tag{39}$$

Substituting (26) into (39) yields

$$p_{11}^{(1)}(n) = (u_n + \Delta b_n)(v_n + \Delta c_n) = \hat{u}_n\hat{v}_n.$$

The proof is completed. □

Now let us consider the time part of the Lax pair for (2)

$$y_{nt} = V_n^{(0)}y_n \qquad V_n^{(0)} = \begin{pmatrix} \frac{1}{2}\lambda & v_n^{-1} \\ \lambda u_{n-1}^{-1} & -\frac{1}{2}\lambda \end{pmatrix}. \tag{40}$$

Differentiating $\hat{y}_n = T_ny_n$ with respect to t yields

$$\hat{y}_{nt} = \hat{V}_n^{(0)}\hat{y}_n \qquad \hat{V}_n^{(0)} = (T_{nt} + T_nV_n^{(0)})T_n^{-1}. \tag{41}$$

Let the two basic solutions ϕ, ψ of (1) satisfy equation (40) as well. Then we can prove the following assertion.

Proposition 2. The matrix $\hat{V}_n^{(0)}$ defined by (41) has the same form as $V_n^{(0)}$, in which the old potentials u_n, v_n are mapped into $\hat{u}_n, \hat{v}_n$ according to the same DT (34).

Proof. Let $T_n^{-1} = T_n^*/\det T_n$ and

$$(T_{nt} + T_nV_n^{(0)})T_n^* = \begin{pmatrix} g_{11}(\lambda, n) & g_{12}(\lambda, n) \\ g_{21}(\lambda, n) & g_{22}(\lambda, n) \end{pmatrix}. \tag{42}$$

Through direct calculations, we know that $\lambda^{-1}g_{11}, \lambda^{-1}g_{21}, \lambda_{22}^{-1}$, and g_{12} are quadratic polynomials in λ. By virtue of (36), (23) and (41), we have

$$\begin{aligned} &\lambda_i a_{nt} = -b_{nt}\alpha_i(n) - b_n\alpha_{it}(n) \\ &d_{nt} = -c_{nt}\alpha_i^{-1}(n) + c_n\alpha_i^{-2}(n)\alpha_{it}(n) \\ &\alpha_{it}(n) = \lambda_i u_{n-1}^{-1} - \lambda_i\alpha_i(n) - v_n^{-1}\alpha_i^2(n). \end{aligned} \tag{43}$$

It is easily verified by (36) and (43) that λ_1 and λ_2 are roots of $g_{ij}(\lambda, n), i, j = 1, 2$. Therefore we have

$$(T_{nt} + T_n V_n^{(0)}) T_n^* = (\det T_n) \begin{pmatrix} \lambda q_n^{(11)} & q_n^{(12)} \\ \lambda q_n^{(21)} & \lambda q_n^{(22)} \end{pmatrix}$$

that is

$$(T_{nt} + T_n V_n^{(0)}) = \begin{pmatrix} \lambda q_n^{(11)} & q_n^{(12)} \\ \lambda q_n^{(21)} & \lambda q_n^{(22)} \end{pmatrix} T_n \tag{44}$$

where $q_n^{ij} (i, j = 1, 2)$ are independent of λ. Comparing the coefficients of λ^2 and λ in (44), we get

$$q_n^{(11)} = \tfrac{1}{2} \qquad q_n^{(22)} = \tfrac{1}{2} \qquad q_n^{(21)} = \frac{1}{a_n u_{n-1}} (u_{n-1} c_n + d_n) \qquad q_n^{(12)} = \frac{a_n - b_n v_n}{d_n v_n} \tag{45}$$

which together with (27, 28) imply

$$q_n^{(21)} = \frac{1}{u_{n-1} + \Delta b_{n-1}} = \frac{1}{\hat{u}_{n-1}} \qquad q_n^{(12)} = \frac{1}{v_n + \Delta c_n} = \frac{1}{\hat{v}_n}. \qquad \square$$

Since the differential-difference equation (2) is equivalent to the discrete zero-curvature equation, $U_{nt} + U_n V_n^{(0)} - V_{n+1}^{(0)} U_n = 0$, from propositions 1 and 2 we have the following.

Proposition 3. Every solution (u_n, v_n) of (2) is mapped into a new solution $(\hat{u}_n, \hat{v}_n)$ of (2) under the DT (34).

In what follows, we shall apply the DT to give explicit solutions of (2). Substituting the trivial solutions, $u_n = v_n = 1$, of (2) into (1) and (40), two real basic solutions ϕ_n, ψ_n are chosen as

$$\beta^n \exp\left(\frac{t}{2}\sqrt{\lambda^2 + 4\lambda}\right) \begin{pmatrix} 2 \\ \sqrt{\lambda^2 + 4\lambda} - \lambda \end{pmatrix}$$
$$\beta^{-n} \exp\left(-\frac{t}{2}\sqrt{\lambda^2 + 4\lambda}\right) \begin{pmatrix} 2 \\ -\lambda - \sqrt{\lambda^2 + 4\lambda} \end{pmatrix}$$

with $\beta = \frac{1}{2}(2 + \lambda + \sqrt{\lambda^2 + 4\lambda}), \lambda \in I_0 = (-\infty, -4) \cup (0, +\infty)$. Noticing (22) and (23), new explicit solutions of (2) are obtained with the help of the DT (34):

$$\begin{aligned} \hat{u}_n &= 1 + \Delta \frac{(\beta_1^{2n} e^{\delta_1 t} - \gamma_1)(\beta_2^{2n} e^{\delta_2 t} - \gamma_2)}{\xi_1 + \xi_2 \beta_1^{2n} e^{\delta_1 t} + \xi_3 \beta_2^{2n} e^{\delta_2 t} + \xi_4 (\beta_1 \beta_2)^{2n} e^{(\delta_1 + \delta_2) t}} \\ \hat{v}_n &= 1 + \Delta \frac{(\beta_1^{2n} e^{\delta_1 t} + \eta_1)(\beta_2^{2n} e^{\delta_2 t} + \eta_2)}{\zeta_1 + \zeta_2 \beta_1^{2n} e^{\delta_1 t} + \zeta_3 \beta_2^{2n} e^{\delta_2 t} + \zeta_4 (\beta_1 \beta_2)^{2n} e^{(\delta_1 + \delta_2) t}} \end{aligned} \tag{46}$$

where

$$\delta_i = \sqrt{\lambda_i^2 + 4\lambda_i} \qquad \beta_i = \beta(\lambda_i) \qquad \eta_i = \frac{\gamma_i(\delta_i + \lambda_i)}{\delta_i - \lambda_i} \qquad i = 1, 2$$
$$\xi_1 = \frac{\gamma_1 \gamma_2 (\delta_1 \lambda_2 - \delta_2 \lambda_1)}{2(\lambda_2 - \lambda_1)} \qquad \xi_2 = \frac{\gamma_2 (\delta_1 \lambda_2 + \delta_2 \lambda_1)}{2(\lambda_2 - \lambda_1)}$$
$$\xi_3 = \frac{\gamma_1 (\delta_1 \lambda_2 + \delta_2 \lambda_1)}{2(\lambda_1 - \lambda_2)} \qquad \xi_4 = \frac{\delta_1 \lambda_2 - \delta_2 \lambda_1}{2(\lambda_1 - \lambda_2)}$$
$$\zeta_1 = \frac{2\lambda_1 \lambda_2 \gamma_1 \gamma_2 (\delta_1 - \delta_2 + \lambda_1 - \lambda_2)}{(\lambda_1 - \lambda_2)(\delta_1 - \lambda_1)(\delta_2 - \lambda_2)} \qquad \zeta_2 = \frac{2\lambda_1 \lambda_2 \gamma_2 (\delta_1 + \delta_2 - \lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)(\delta_1 - \lambda_1)(\delta_2 - \lambda_2)}$$
$$\zeta_3 = \frac{2\lambda_1 \lambda_2 \gamma_1 (\delta_1 + \delta_2 + \lambda_1 - \lambda_2)}{(\lambda_2 - \lambda_1)(\delta_1 - \lambda_1)(\delta_2 - \lambda_2)} \qquad \zeta_4 = \frac{2\lambda_1 \lambda_2 (\delta_1 - \delta_2 - \lambda_1 + \lambda_2)}{(\lambda_2 - \lambda_1)(\delta_1 - \lambda_1)(\delta_2 - \lambda_2)}.$$

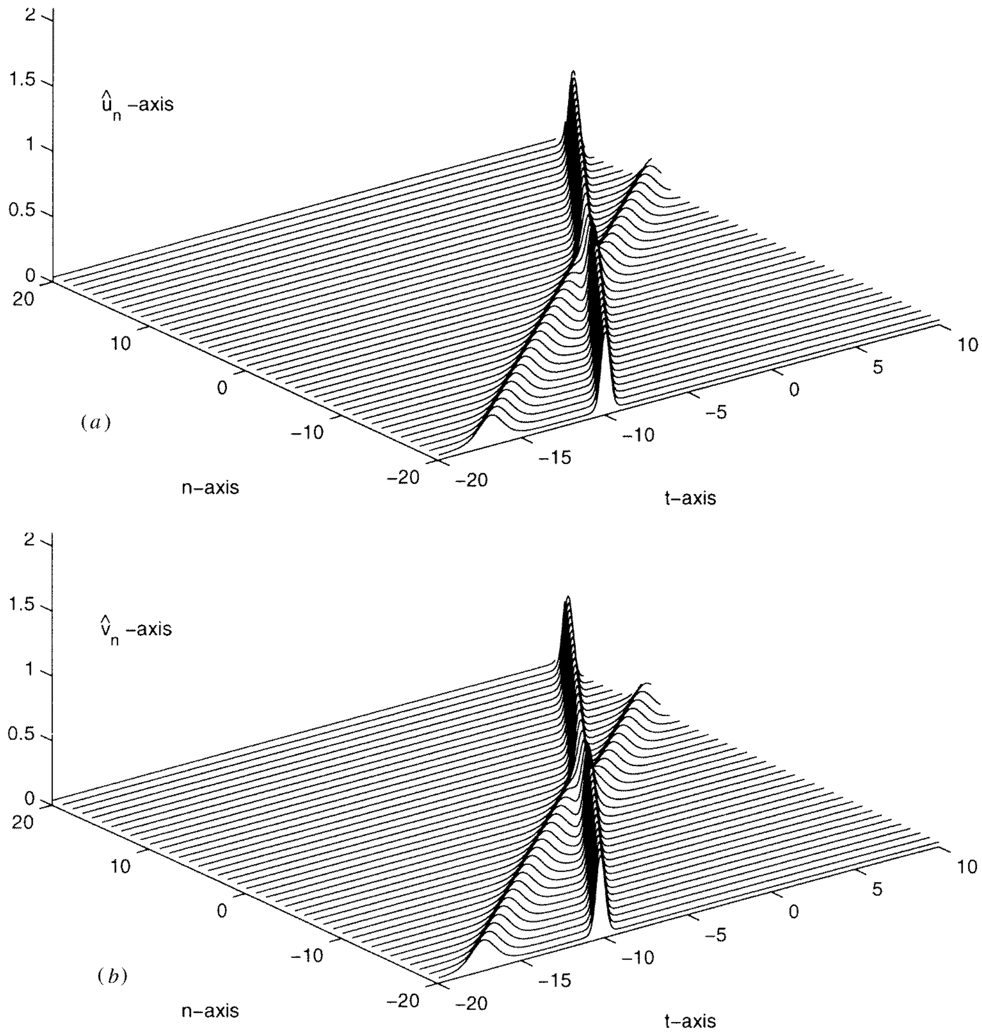

Figure 1. The two-soliton solutions with $\gamma_1 = -3$, $\gamma_2 = 5$, $\lambda_1 = -5$, $\lambda_2 = -10$.

The solutions (46) are two-soliton solutions if the parameters are chosen as $\gamma_1 = -3$, $\gamma_2 = 5$, $\lambda_1 = -5$, $\lambda_2 = -10$ (see figure 1). Further, if the two-soliton solutions are taken as the new starting point, we can make the DT once again by (34) and engender another set of new solutions. This process can be done continually and will usually yield a series of multi-soliton solutions.

This work was partially supported by Hong Kong RGC/97-98/21. Project 19671074 was supported by National Natural Science Foundation of China. XGG would like to thank the Henan Science Foundation Committee of China for financial support. We would like to thank Professor Zhu Siming for his help in drawing the figure by computer.

References

[1] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[2] Schilling R J 1989 *J. Math. Phys.* **30** 1487–501
[3] Tu G Z 1990 *J. Phys. A: Math. Gen.* **23** 3903
[4] Merola I, Ragnisco O and Tu G Z 1994 *Inverse Problems* **10** 1315
[5] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)
[6] Matveev V B and Salle M A 1991 *Darboux Transformation and Solitons* (Berlin: Springer)
[7] Levi D 1988 *Inverse Problems* **4** 165
[8] Gu C H and Zhou Z X 1994 *Lett. Math. Phys.* **32** 1